Forecasting Private Equity Capital Calls using Beta Regression

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Introduction

UTIMCO is an external investment company that manages the endowment for the University of Texas and Texas A&M systems. Currently, the fund has over $43.2 billion in assets under management [1], an important portion of which is invested in Private Equity (PE) vehicles. The aim of this project is to use predictive methods to improve on the model currently used by UTIMCO investment managers to forecast cash calls from these PE investments. Having quick, reliable, and actionable predictions of outflows of cash will allow investment managers to more accurately evaluate the condition of their investments and to consider additional investment opportunities. Looking forward, UTIMCO may explore ways to implement similar models in other parts of their investment portfolio.

Investment planning in the face of uncertain future capital positions is ubiquitous in the investment industry. The PE arena is no exception. When large institutional investors seek exposure to private investments not present in public markets, often times they enter into a relationship with a PE fund. The fund requires the investors to commit some amount of capital that can be called at the fund's discretion as new investment opportunities arise. These calls regularly amount to huge sums and are uncertain to the investor. The classic dilemma an investment manager faces is to find a balance between over-exposing themselves to the risk of being unable to meet future capital calls (over-investing) and leaving revenue-generating opportunities unfunded (under-investing). An accurate predictive model will allow a manager to walk this fine line. Our approach is to forecast cash calls using a machine-learning approach. We propose a beta regression that uses historical time-series data to forecast contribution rates (CRs) for specific time periods. From there, we create a period ahead portfolio level forecast of capital calls.

Model Specifications

We assume that for each investment, there is a pre-defined and unchangeable commitment amount, C. This represents the total dollar amount the investor has agreed to contribute to the PE fund. The cumulative capital calls up until period t is defined as F_t. For each time period, the following identity must always hold:

$$(C - F_t) \geq CapitalCall_t$$

Equation (1)

Where $CapitalCall_t$ is the dollar amount of capital called at each time period. Intuitively, this equation states that a private equity fund cannot call more than the total remaining capital at any time period. We model cash flows in time period, t, using the following equation:

$$\alpha_t(C - F_t) = CapitalCall_t$$

Equation (2)

[1] https://www.utimco.org/funds/AllFunds/assetsundermngt.pdf

α_t is interpreted as the percent of the remaining commitment outstanding that is called at t.As a PE fund cannot call more than 100% of remaining commitment with this formulation, α_t must lie between 0 and 1. We refer to α_t as the contribution rate (CR). This cash flow model[2] is desirable for two main reasons. First, it has an asymptotic nature, implying that most of the capital will be called in the earlier periods. The relative magnitude of capital calls will decrease in later periods. We can use various α parameters to visualize the impact of a fixed contribution rate on per period cash flows over the life of a fund.

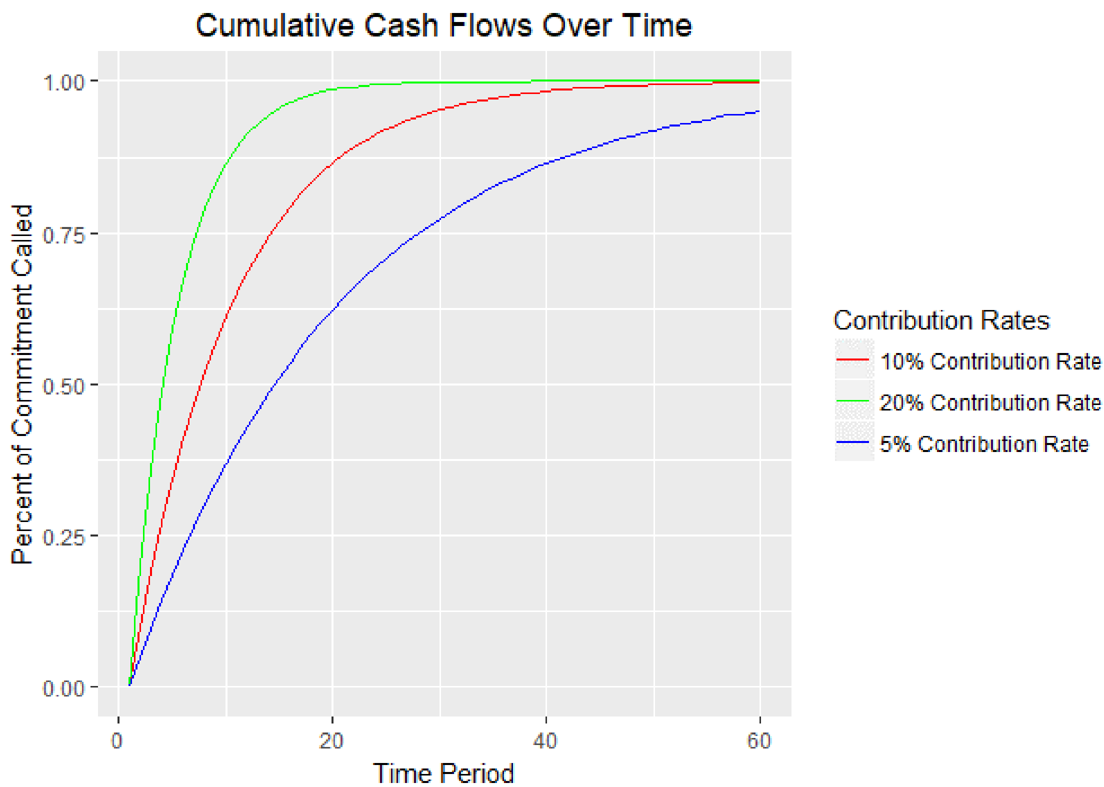

Figure 1: Hypothetical investment capital call with differing contribution rates (α_t)

Second, we can calculate an empirical contribution rate per period per investment in the historical data. We rearrange Equation (2) to solve for the CR.

$$\alpha_t = \frac{CapitalCall_t}{(C - F_t)}$$

Equation (3)

When implementing regression methods in forecasting problems, the model must be fit to data which are independent and identically distributed (*iid*). Historical data of portfolio positions are unsuitable as there will be a high degree of autocorrelation. These data must be transformed in a way that allows the *iid* assumption to hold. Calculating CRs and using them to train the model is a reasonable solution. Philosophically, this is similar to converting equity prices into returns to aid in calculation and comparison.

[2]For more details on this equation see Takahashi, Alexander (2001)

In practice, however, the rate of contribution is not fixed from period to period as it is in Figure 2. This rate is impacted by a variety of factors, including macroeconomic conditions and fund level characteristics. Buchner, Kaserer, and Wagner (2009) described the contribution rate as a mean reverting stochastic process. We present a novel approach to modeling the contribution rate per period in a discrete time environment.

Beta Regression: Modeling Contribution Rate

We begin by assuming that each contribution rate per period, per investment is an independent draw from an unknown beta distribution[3]. While we could build a model using a single, portfolio-level beta distribution based only on historical CRs, a better decision would be to fit a beta distribution conditioned upon upon a series of fund features. Therefore, we can define a beta regression model that allows for this conditioning for the contribution rate at time t.

Beta regression falls into the class of generalized linear models and is commonly used to measure rates and proportions. The regression, much like the beta distribution, constrains outputs between 0 and 1 and can assume a variety of flexible shapes (See Appendix 1 for examples of different beta distributions). A beta distribution is usually described by two parameters α (not to be confused with the α previously defined as the contribution rate) and β. We follow Ferrari and Cribari-Neto's (2004) decision to reparameterize[4] the beta distribution into the more interpretable μ - the expected value - and ϕ - the precision. We then use the beta regression as defined by Simas et. all (2010) which allows us model the μ and ϕ for each observation.[5] In the context of private equity investing, our response variable is the contribution rate for period t for investment k, $\alpha_{k,t}$. Keeping with the standard notation in the academic literature, we will denote $\alpha_{k,t}$ as μ_i. Furthermore, the precision parameter for an individual fund at period t can be written as $\phi_{k,t}$. For simplicity, we denote $\phi_{k,t}$ as ϕ_i. We describe the following model for μ_i [6].

$$\mu_i * = B_0 + B_1 log(TimePeriod_t) + B_2 LagCumulativePercentage + B_3 VehicleStructureFund + \epsilon$$

Equation (4)

$$\hat{\mu_i} = g^{-1}(\mu_i^*)$$

Equation (5)

$B_{0\ldots3}$ are the coefficient terms for the regression with B_0 acting as the intercept. $log(TimePeriod_t)$ is the natural log of t, the forecasting period. This variable is meant

[3] This independence assumption is important as it allows statistically rigorous analysis. Furthermore, the historical data suggests that there is no statistically significant autocorrelation between contribution rate across a particular fund's life.

[4] This reparameterization of the beta distribution allows us to better interpret the parameters of the beta distribution. μ is the expected value of the distribution while ϕ is a measure of spread. From the traditional parameterization using p and q to μ and ϕ is straightforward. $\mu = \alpha/(\alpha + \beta)$ and $\phi = \alpha + \beta$.

[5] This per observation distribution allows us to forgo the traditional homoscedastic (constant variance) assumption for residuals present in ordinary least squares(OLS) linear regression.

[6] This is the variable dispersion beta regression model as defined by Simas et. all (2010)

to capture the gradual changes in contribution rates as the fund progresses. $t = 1$ is the first period at which capital was called by the private equity fund. $t = 2$ is the period after the first period, which in the empirical data set, may or may not host a capital call. $LagCumulativePercentage$ is the cumulative percentage of capital called up until $t - 1$. For example, if there was an investment that had called \$80 up until the forecasting period out of a \$100 commitment amount, $LagCumulativePercentage$ would be $80/100 = 0.8$. $VehicleStructureFund$ is a dummy variable indicating whether the PE vehicle pertains to a Fund or Co-Investment. g^{-1} is a link function transforming the linear output into a standardized value between zero and one. In this model, we used the logit function as our link function for μ_i.[7]

Furthermore, we define the following model for ϕ_i:

$$\phi_i* = \zeta_0 + \zeta_1 log(TimePeriod_t) + \zeta_2 VehicleStructureFund + \epsilon$$

Equation (6)

$$\hat{\phi_i} = f^{-1}(\phi_i*)$$

Equation (7)

Where $\zeta_{0...2}$ are the coefficients of the model with ζ_0 acting as an intercept term. $VehicleStructureFund$ and $log(TimePeriod_t)$ retain the same definitions as before. f^{-1} is another link function transforming ϕ_i* into $\hat{\phi_i}$. In this formulation, we define f^{-1} as the natural log. Finally, by allowing us to model ϕ_i and μ_i, the beta regression implies a variance of the distribution from which μ_i was generated. Thatvari ance is calculated as follows:

$$\hat{Var}(\mu_i) = \frac{\mu_i(1-\mu_i)}{1+\phi_i}$$

Equation (8)

These models' parsimony (using a small number of regressors) deserves a mention. In extremely noisy environments like financial markets, forecasting is an inherently difficult problem. It is not unusual for year ahead forecasts to be off by as much as 100%, especially when calibrating a model to historical data. We combat this issues by minimizing the number of regressors. While it does make the model more inaccurate in back tests, the parsimony increases the model's robustness in real world applications by preventing over fitting to historical data.

Portfolio Level Aggregation - Expected Cash Flow

As large institutional investors typically invest in many funds, it is important to have a mechanism for aggregating the investment level contribution rate forecasts into a portfolio level capital call projection. We calculate a contribution rate forecast for the portfolio for time period, t by taking the weighted average of the investment level contribution rate forecasts. These predicted contribution rates are weighted by $C_k - F_{t,k}$, or the commitment outstanding for each investment,k.

[7] The logit function is a commonly used link function and takes the following form: $x = e^{x*}/(1 + e^{x*})$.

From this weighted average contribution rate, M, we can calculate the predicted dollar level cash flows for the portfolio for the upcoming period by summing up the commitment outstanding for all investments and multiplying that value by M. This can also be expressed with the following equation:

$$Total\hat{Capital}Calls_t = M \sum_{k=1}^{K} (C_k - F_{t,k})$$

Equation (9)

Portfolio Level Aggregation - Expected Cash Flow Variance

It is important that we calculate confidence bounds in order for this analysis to aid in real-world decision-making. Recall that the variance of the sum of two random variables, X and Y can be calculated using the following formula:

$$Var(X + Y) = Var(X) + Var(Y) + 2Cov(X, Y)$$

Equation (10)

If X and Y are scaled by factors a and b,

$$Var(aX + bY) = a^2Var(X) + b^2Var(Y) + 2abCov(X, Y)$$

Equation (11)

If we view μ_i as the expected value of a random variable and $Var(\mu_i)$ as the estimated variance of the predicted beta distribution, we can use this concept to calculate the estimated variance of the weighted portfolio level contribution rate, M

$$Var(M) = w_1^2Var(\mu_1) + w_2^2Var(\mu_2) + ... + w_i^2Var(\mu_i)+$$
$$2w_1w_2Cov(\mu_1, \mu_2) + 2w_1w_3Cov(\mu_1, \mu_3) + ... + 2w_kw_{k-1}Cov(\mu_i, \mu_{i-1})$$

Equation (12)

Where w_k is the percentages of investment k's commitment outstanding over the total portfolio commitment outstanding. In practice due to data sparsity issues, it would be nearly impossible to generate an accurate covariance matrix between CRs among funds. For simplicity, we assume the covariance value between CRs among funds is .1. This is a conservative estimate and seems to be correspond reasonably well with historical data.

Portfolio Confidence Bounds

Now that we have the variance for M, we can calculate portfolio level confidence bounds for all capital calls in time period, t. Where UCB is the upper confidence bound for M (the weighted predicted contribution rate for the portfolio) and where LCB is the lower confidence bound for M.[8]

$$UCB = M + 2\sqrt{Var(M)}$$

Equation (13)

$$LCB = M - 2\sqrt{Var(M)}$$

Equation (14)

Preprocessing Details

In order to evaluate model performance, we split the historical transactions into two separate data sets: an in-sample set, which was used to "train" the model's coefficients and an out of sample set, which was used to evaluate the model's performance. While back-testing, we created a sliding window for our train set. We only included data from $t-10$ to $t-1$ when we were attempting to forecast t. In other words, if we were forecasting capital calls on January 1st, 2010, we would look at historical capital calls for all investments from January 1st 2000- December 31st 2009. We created this sliding windows to drop capital calls from decades back which would not have any predictive power.

Furthermore, we recognize that this forecasting method does not attempt to forecast new investments which will be made by the fund over the year, simply the capital calls of existing PE investments. Therefore, in order to get improved coefficient estimates, we have to normalize the in-sample ("train") and out-of-sample ("test") data. This was conducted by eliminating capital calls which occurred at $t = 1$, the initial capital call, from our training set.

Finally, our data set contained many contribution rates that were 0 or 1. These extreme values do not lie within any beta distribution and must be dealt with. We performed the transformation employed by Smithson and Verkuilen (2006) [9] to coerce the data into the domain.

Model Output

We forecast year ahead capital calls using historical transaction level. Iterating year by year, we evaluated our model from 2010 to 2017. Sitting at January 1st of each year, we look at the investments still calling capital, forecast expected contribution rates, and then calculate portfolio capital calls for the year ahead. This resulted in the following predictions:

[8] We are not able to interpret these bounds with traditional confidence levels. If you wish to calculate the confidence level, you integrate the joint density function of all the predicted beta distributions for each fund's CR to a particular level.

[9] The Smithson transform is defined as the following : $\frac{x(n-1)+.5}{n}$ where x is the original response value and n is the sample size.

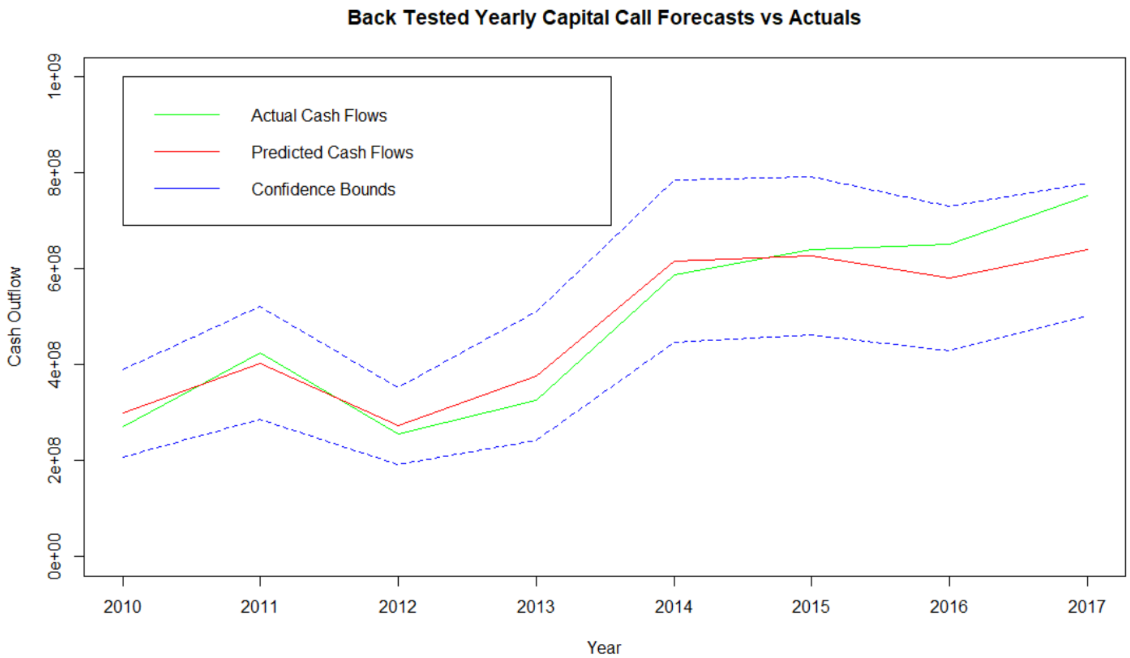

Figure 2: Predicted vs. Actual Capital Calls for the years 2010 to 2017 with confidence bounds based upon a .1 covariance value between all funds.

These back-tests generated the following error metrics.

Error Metric	Value
Mean Error (ME)	$12,005,530
Mean Absolute Error (MAE)	$43,204,377
Mean Percentage Error (MPE)	.53%
Mean Absolute Percentage Error (MAPE)	8.87%

Table 1: Forecast errors for beta regression based model from 2010 to 2017

For a comprehensive documentation of model assumptions, see Appendix 2. From the beta regression model defined in Equations 4, 5, 6, 7, we fit the model parameters $B_{0...3}$ and $\zeta_{0...2}$ using Maximum Likelihood Estimation. These coefficients along with the corresponding t-tests for the January 1st, 2017 back test are in Appendix 3 for μ_i and in Appendix 4 for ϕ_i.

Economic Value of Model

The economic value of this model stems from the firm's ability to shift the proportion of their capital in liquid investments to PE investments. By reducing the uncertainty in expected future cash positions, the investment company will be able to allocate capital more heavily in PE vehicles without increasing the risk of not being able to meet cash calls. This shift will result in the firm capturing higher expected rates of return on a portion of their assets invested in low-return vehicles.

To illustrate on how the economic value calculations were performed, we create two probability density plots. Figure 3 shows hypothetical distributions of the errors of the two models (a previously used model and an improved model). Note that in this hypothetical case, both distributions have the same expected values with different standard deviations. The improved model would be the one with a smaller standard deviation, hence less uncertainty.

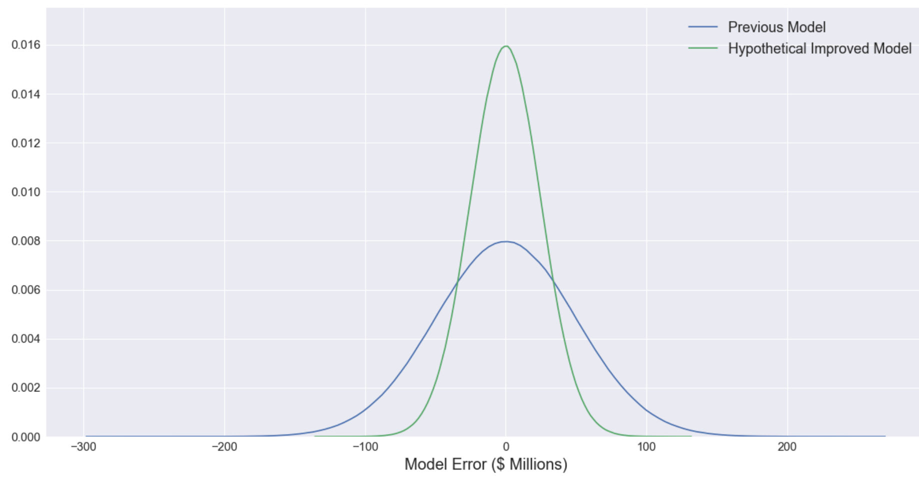

Figure 3: Hypothetical Model Error

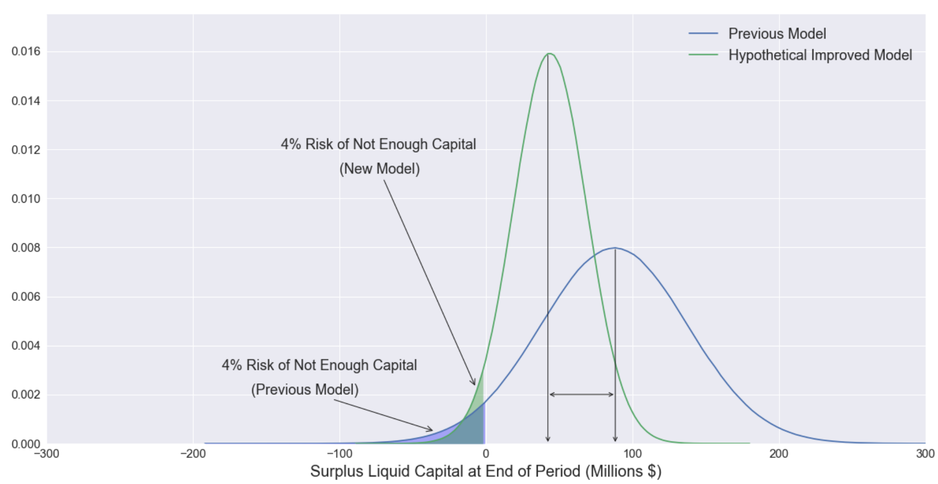

Figure 4: Hypothetical Surplus Reserves for Capital Calls

Figure 4 shows distributions of surplus capital at the end of an investment period. If the resulting surplus capital is above zero, then this means that the investment company reserved excess liquid capital for the calls made during that period, which could have been invested elsewhere. If the resulting surplus capital is below zero, this means that the investment company did not have enough liquid capital to meet all the calls made during that period. Both of these situations are sub-optimal. The ideal scenario would be one where a period ended with zero surplus liquid capital.

Given a fixed risk tolerance for the probability of not being able to meet all calls, an improved model with a lower standard deviation would require lower capital reserves when compared to some previous model. This means that with an improved model, the investment company can reduce its excess capital level from the mean of the current model's distribution to the mean of the distribution of the new model. Therefore, the yearly economic value of our model value becomes a function of the variance of the model's errors and is calculated using the following equation:

$$\frac{ExcessCapital_f - ExcessCapital_n}{1+r}$$

Equation (15)

Where $ExcessCapital_f$ is the reserve capital allocation for the previous model, $ExcessCapital_n$ is the reserve capital allocation for the improved model, r is the opportunity cost of capital (discount rate).

We assume that the investment company's risk tolerance for not having enough capital is 4%, any capital deployed in a PE investment will return an expected 6% per year, and that both model errors are normally distributed. Furthermore, we assume that $750 MM will be called for existing investments during the following year and that the original model has a MAPE of 10%, ME of 0% and Standard Error of 12.6%. With these assumptions, $165.8 MM will be needed as excess capital so that there is only a 4% risk of not having enough capital. In contrast, with the new model (MAPE 8.9%; ME -0.5%; Standard Error 10.6%) we calculate that that $139.6 MM will be needed as excess capital for the same level of risk. With a 6% annual return, this reduction in excess capital of $26.3 MM could be yielding $1.58 MM per year. Treated as an annuity, the present value of these savings would be worth $26.3 MM to the firm.

Conclusion

In this paper, we proposed a novel method to forecast private equity capital calls. Using historical transaction data, we fit a beta regression model on per period contribution rates for many investments. This model performed well on year ahead forecasts for the years 2010 to 2017, resulting in a mean percentage error of -.5% and a mean absolute percentage error of 8.9%. We also defined a framework to calculate the financial value of private equity forecast models for institutional investors and estimated the value of this new model.

References

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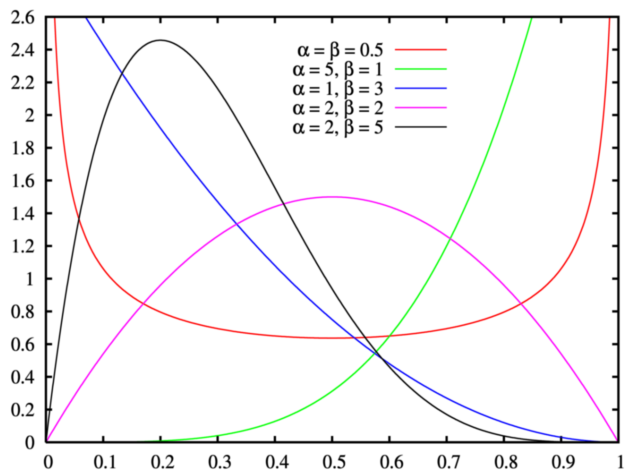
Appendix

Appendix 1



Source: Wikipedia

Possible shapes of the beta distribution probability density function

Appendix 2

Beta Regression and Forecast Model Assumptions

1. The contribution rate at each time, t , a_t must be between 0 and 1.
2. For each time period, the PE fund can never call more capital than is remaining, $C - F_t$.
3. The institutional investor will know which funds have the potential to call capital in the upcoming period.
4. Portfolio call level forecasting consists of commitments already made and excludes new investments that have not yet begun.
5. The covariance for the contribution rates among funds is .1. As mentioned before, we believe that this is a reasonable number.

Appendix 3

	Estimate	Standard Error	z value	p value
Intercept	-1.07	0.28	-3.90	9.56e-5
log(Time Period)	-0.63	0.12	-5.20	3.30e-7
lagCumPercentCalled	1.84	0.19	9.79	<2.16e-16
VehicleStructureFund	0.49	0.26	1.90	5.74e-2

Table 2: Beta Regression Coefficient Estimates for 2017 Forecast for μ_i with logit link

Appendix 4

	Estimate	Standard Error	z value	p value
Intercept	1.49	0.24	6.09	1.11e-9
log(Time Period)	-1.36	0.09	-15.23	<2.16e-16
VehicleStructureFund	1.41	0.22	6.44	1.23e-10

Table 3: Beta Regression Coefficient Estimates for 2017 Forecast for ϕ_i with log link